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Exploring the Division Algorithm in Euclidean Domains with Exploding Dots

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by

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ABSTRACT

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We will give an overview of the representation of place value and arithmetic known as Exploding Dots and use this idea to explore the division algorithm. It is well-known that the ring of integers, the ring of polynomials, and the ring of Gaussian integers are all examples of Euclidean domains and therefore possess a division algorithm. Exploding Dots beautifully illustrates how one can perform division in any base and how this naturally leads us to division of polynomials. We will show how this same idea of having a "base machine" can be used to perform division in the Gaussian integers. No prior knowledge is assumed, and anyone can play and be immersed in the realm of Exploding Dots.

Thank You.

Professor Kevin McLeod for your abundant support, guidance, and time in steering me to the completion of this thesis.

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Nicholas Johnson

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Introduction

I.1 An intro to Exploding Dots

Exploding Dots is, at its core, the story of place value in Mathematics. It was developed by mathematician and founder of the Global Math Project, James Tanton. James is possessed by the conviction that mathematics can be made understandable to, and Exciting for, everyone. Regardless of one's background in mathematics or their current relationship with the subject, Exploding Dots is an invitation to all to have an experience of uplifting joy and understand ideas in a whole new light. We will give a brief introduction on how this "dots-and-boxes" method can be used to visualize different base representations and perform all basic arithmetic operations in a very conceptual albeit non-standard way. We will focus primarily on division though, and show how Exploding Dots beautifully illustrates how we can perform division in any base, and how this naturally leads us to division of polynomials. We will then define a Euclidean Domain and be reminded that the ring of integers and the ring of polynomials are examples of Euclidean Domains. We will then show how this same idea of having what will will call a base machine can be used to perform division in the Gaussian Integers.

The story of Exploding Dots begins with playing with what we'll call a $1 \leftarrow 2$ machine. We will pronounce this as a two-to-one machine. This machine is nothing more than a row of boxes where we can put as many dots as we like into our rightmost box. This machine has a rule that whenever there are two dots in any box, these dots explode and become one dot one box to the left. So if we just begin to count we find that every natural number has a unique code in this machine.

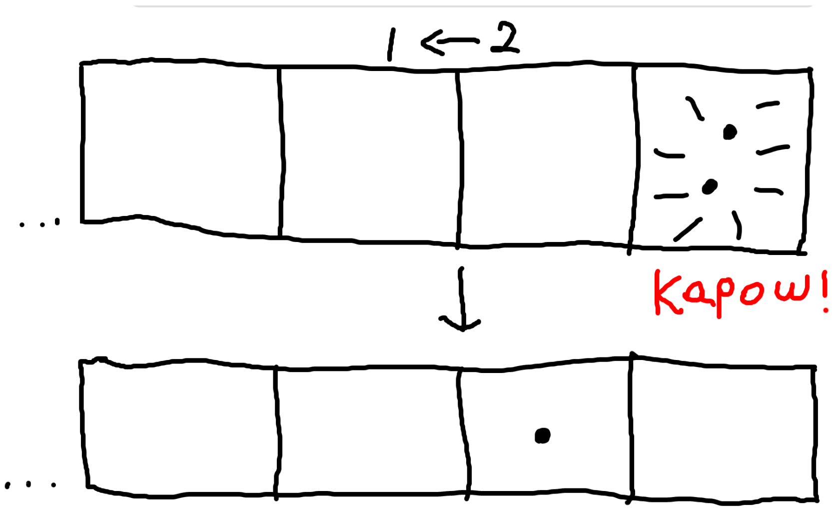


Figure I.1: $1 \leftarrow 2$ machine

If we put in one dot nothing happens, so the code for one is 1. But as is illustrated in I.1 if we put in a second dot we have an explosion, and then we are left with one dot in our second box and no dots in the last box. So we read the code for two as 10. Yes, these $1 \leftarrow 2$ machine codes are the base 2, or binary, representation of numbers. However, there are no pre-requisites to play with this machine. Anyone, regardless of their mathematical background can play with this machine and discover "codes" for numbers. It then becomes natural to determine what the value of each box must be. The machine is set up that we always place dots in the rightmost box, so each dot in this first box must be worth 1. But then two 1s make one dot one space over. So therefore each dot in the second box must be worth 2. And two 2s make one in the next box, so each dot in the third box must be worth 4. And two 4s make one of the next and so on. So we can easily see that we have our boxes as powers of 2. This story continues with having a flash of insight. Instead of having a $1 \leftarrow 2$ machine, we could play with a $1 \leftarrow 3$ machine! Same type of machine, but now whenever there are 3 dots in a box they explode away to be replaced with one dot one box to the left. And you could easily create codes for natural numbers in this machine. But then you might have another flash of insight and realize that you could play with a $1 \leftarrow 4$, a $1 \leftarrow 5$, or a $1 \leftarrow 6$ machine. How about let's go crazy and go all the way to a $1 \leftarrow 10$ machine! And let's go really crazy and put 273 dots in this machine! What is the secret $1 \leftarrow 10$ code for 273? This may seem like a very simple idea. And it is! Students of any age or ability level can have genuine fun playing with these machines and get a true sense of understanding of different base representations. It is actually strange that we have words in English for "eleven" and "twelve" albeit we don't have symbols to represent these numbers. Wouldn't onety-one and onety-two make more sense than "eleven" and "twelve?" Now it should be obvious that we can have a $1 \leftarrow b$ machine where we would use "digits" $0, 1, 2, ..., b-1$ and this will put us in base b. Notice that our representation is digit or symbol independent. However, we can play with these machines and find ourselves in the world of fractional bases where we actually end up using "digits" that are greater than the base.

A $2 \leftarrow 3$ machine represents numbers in base $\frac{3}{2}$, but not quite in the "usual" mathematical meaning of a base representation: in our Exploding Dots scenario the digits 0, 1, and 2 are allowed. This might seem strange that we're using a digit that is larger than the base. But we really are in base one-and-a-half here as Exploding Dots clearly shows that every natural number has a unique representation in this $2 \leftarrow 3$ machine. More generally, for a $q \leftarrow p$ machine where $p > q$, the digits 0,1,..., p-1 are allowed. We call this the $\frac{p}{q}$ representation of a number. All positive integers can be written in the form $a_n b^n + a_{n-1} + ... + a_1 b + a_0$ where $b = \frac{3}{2}$ and each a_k = 0,1 or 2. This is called the sesquinary (for "one and a half") representation of a positive integer. Exploding Dots, namely the $2 \leftarrow 3$ machine shows that this is possible. There are several interesting and open problems in base $\frac{3}{2}$ that could be fun to explore, for example, it is obvious that every natural number has a unique representation in the $2 \leftarrow 3$ machine but there are codes like 201 which corresponds to $\frac{11}{2}$. So if we're given a rational number can we tell if we can write it in base $\frac{3}{2}$ with $0s$, $1s$, or $2s$? Or if we're given some long code can we tell if it's an integer quickly and efficiently without working out all the sums of powers of $\frac{3}{2}$? There are several other open problems as well that you're encouraged to ponder, however we won't be exploring fractional bases in this thesis.

Now if we just stayed in base ten for a little while, Exploding Dots allows one to make genuine sense of all the arithmetic that we typically learn in school. We will call our machine we use in base ten a $1 \leftarrow 10$ machine and this shouldn't cause any confusion. However it should probably be noted that we could very well call this an $1 \leftarrow A$ machine as in a sense every base could be thought of as base "10."

I.2 Addition

Addition is extremely straightforward as we just put two $1 \leftarrow 10$ machines on top of each other and combine dots together. We can go right to left OR left to right. It does not matter! For example if we computed $368 + 187$ we just add our hundreds, add our tens, and add our ones. There's nothing mathematically wrong with an answer of "four-hundred fourteenty-fifteen," it just sounds weird to society. That "ty" is short for ten in English. So we can then take 4|14|15 and perform the explosions. We can do them in any order we choose, to arrive at an answer of five-hundred fivety-five, or as society prefers fifty-five. We could have easily had all dots in our bottom machine pictured here, but chose to just write the digits in each box.

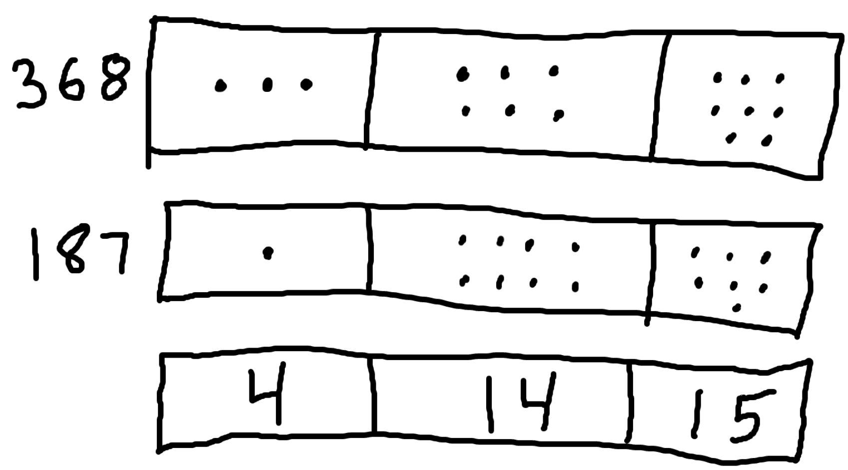


Figure I.2: addition with dots-and-boxes

I.3 Subtraction

Now we are well aware that subtraction is the inverse of addition. In the world of Exploding Dots we choose to believe that subtraction doesn't exist because we can always think of subtraction as adding the opposite. So if we have $612 - 369$ we start with six-hundred onety-two, in other words 612 represented in a $1 \leftarrow 10$ machine and we are adding the opposite of 369, so we therefore need something to represent the opposite of a dot. Let's use an open circle to represent an anti-dot and whenever we have a dot together with an anti-dot together in a box they will annihilate each other.

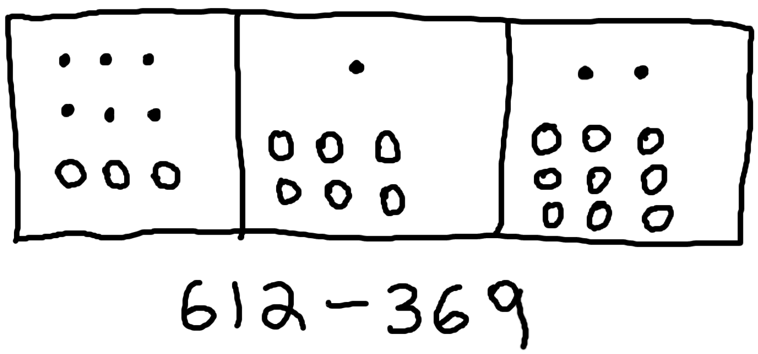


Figure I.3: Subtraction with dots-and-boxes

So if we do the addition we'd have some annihilations and we're left with three hundreds, five anti-tens, and seven anti-ones. We thus have an answer of three-hundred negative five-ty negative seven or $3| - 5| - 7$.

This again is a mathematically correct answer, but if we wanted to fix it up for society's sake we could just unexplode some dots. If we're in a $1 \leftarrow 10$ machine then any dot in a box to the left must have come from ten dots in the box to its immediate right. So we could just unexplode one of our hundreds and we'd have three annihilations in our tens box to give us one-hundred seventy negative-five or $|1||7|| - 5|$. And we could unexplode again which would give us five more annihilations to then give us an answer that society can understand of one-hundred sixty five or 165. The traditional algorithm for subtraction has us work from right

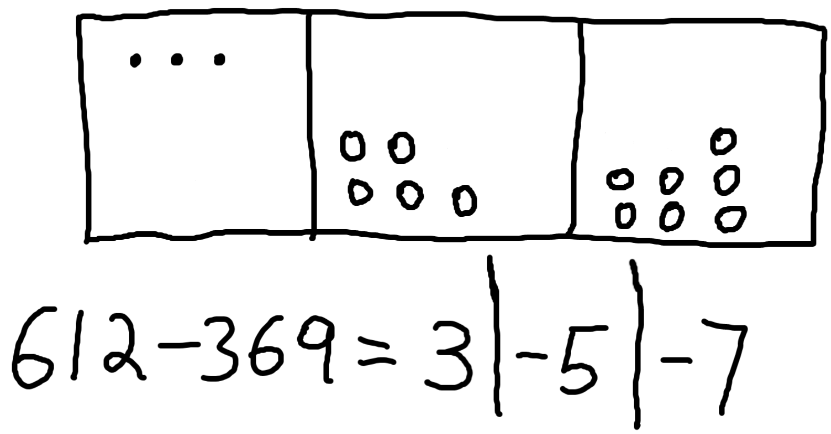


Figure I.4: Mathematically solid answer

to left and do all the unexplosions as we go along. However the Exploding Dots approach just has us go ahead and "do it" and wait until the end to do all of our unexplosions. Both methods are fine to use and correct and it is just a matter of style as to which approach you like best. [KB]

I.4 Multiplication

Multiplication in the Exploding Dots model is extremely straightforward, at least when multiplying any number by one digit. For example, consider $28,613 \times 3$. If we're doing this in base ten, we'd just represent $28,613$ in a $1 \leftarrow 10$ machine and take every dot and triple it.

We can clearly use the same approach in binary or some other base to get a quick answer and then do our explosions afterwards. This method of Exploding Dots also clearly illustrates why when we multiply by b in base b we obtain the original number with a zero tacked on to its end. Let's just look at base ten and consider 28612×10.

So multiplying by ten in base ten will shift all of the digits one place to the left to leave zero dots in the one's place. This will clearly have the same effect multiplying by b if we were in a $1 \leftarrow b$ machine. If we think about what is normally taught about the decimal point

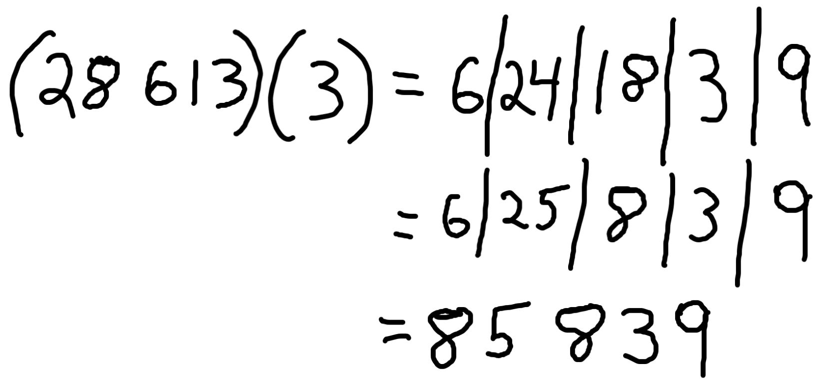


Figure I.5: Just triple everything

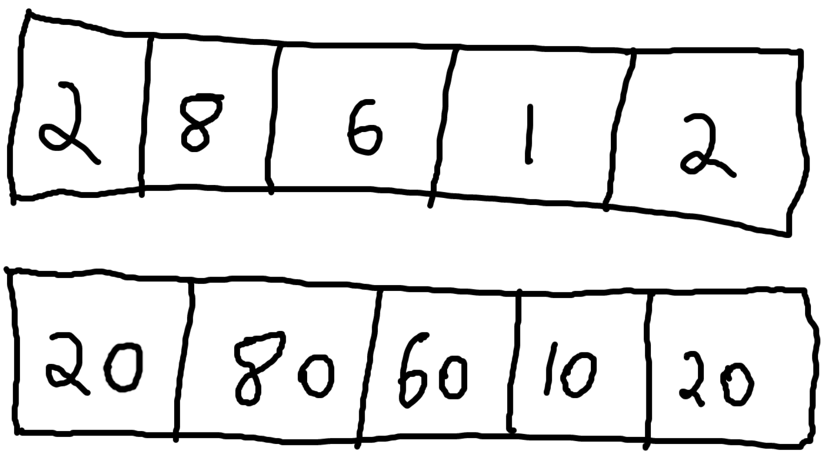


Figure I.6: We'll need one extra box to perform the explosions

moving when multiplying by ten, we see that's not really accurate. What is happening is that the digits are moving, not the decimal point. We can make some sense out of multi-digit multiplication as well, however the dots-and-boxes approach might not be very efficient here. But it certainly is for working with polynomials.

I.5 Polynomials: addition & multiplication

When working with polynomials we will be working in a $1 \leftarrow x$ machine, which works very similar to the machines we have seen, but now we're not saying what base we are in. But as in base two our boxes are powers of two, and in base ten our boxes are powers of ten, in an $1 \leftarrow x$ machine we're in base x so our boxes are powers of x. Adding and subtracting polynomials is no different than in base ten, in fact it is actually easier. In base ten arithmetic we would need to explode dots or perform "carries." But since we don't know the value of x, we don't ever explode dots.

$$\begin{array}{rl} & 6x^2+9x-4 \\ + & 7x^2+7x+5 \\ \hline = & 13x^2+16x+1 \end{array} \qquad \begin{array}{rl} & 9x^3+4x^2-5x+3 \\ - & 2x^3+7x^2+8x-2 \\ \hline = & 7x^3-3x^2-13x+5 \end{array}$$

Figure I.7: We could draw dots in an $1 \leftarrow x$ machine if we'd like

Now consider $(x-2)(3x^2-x+2)$.

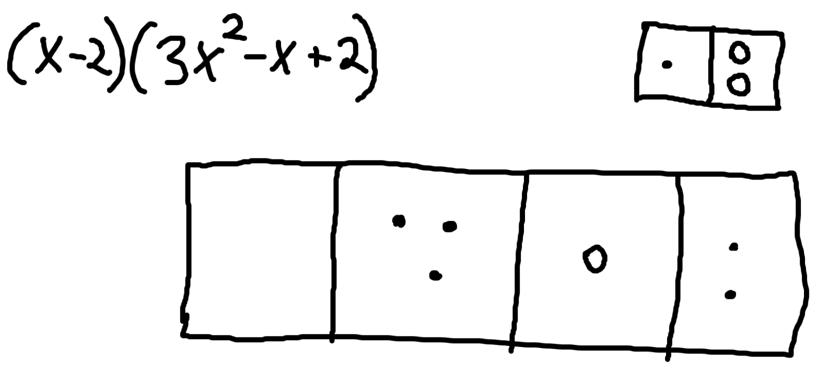


Figure I.8: Just replace with the pattern of our factor

Here we're looking at $3x^2 - x + 2$ in an $1 \leftarrow x$ machine shown in I.8. We know multiplication is commutative so we can multiply here in any order we choose. Since $x - 2$ looks like one dot next to two antidots, if we're multiplying we just replace every dot with one dot next to two antidots or every antidot with the opposite pattern and the solution immediately emerges as we see in I.10.

We note that there is one more operation that we have not discussed yet that we will explore in the next chapter. This gets us to one of the coolest and most conceptual ways

$(x-2)(3x^2-x+2)$

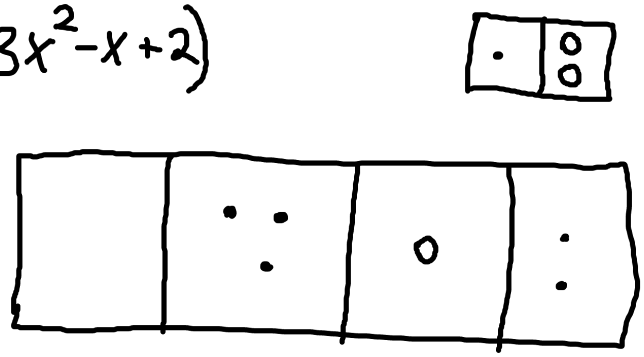
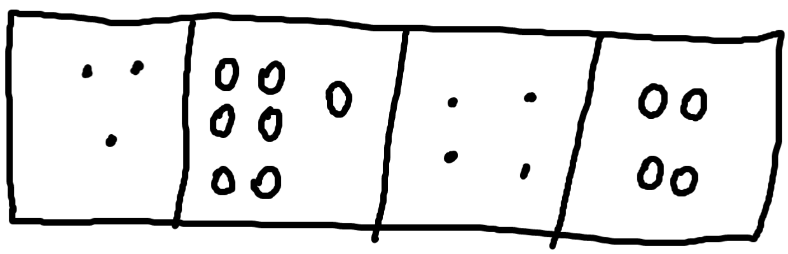


Figure I.9: Just replace with the pattern of our factor



$$= 3x^3 - 7x^2 + 4x - 4$$

Figure I.10: Simple polynomial multiplication

that Exploding Dots can be utilized.

Division

Division can be thought of as repeated subtraction but can also be interpreted as the process of counting groups. Exploding Dots' greatest asset might be in making division exceptionally clear. Let's actually jump straight to polynomial long division! To do so we will be working in our $1 \leftarrow x$ machine, where our boxes are powers of x. Say we wanted to compute $(2x^2 + 7x + 6) \div (x + 2)$. $2x^2 + 7x + 6$ looks like 2 dots, 7 dots, 6 dots. That is 2 dots in our x^2 box, 7 dots in our x box and 6 dots in our 1s box. Now we want to find groups of $x + 2$ which looks like one dot next to two dots. We can just circle the groups in our picture and realize that all of the dots really must be in the rightmost part of our loop. We can see that we have two copies of $x + 2$ at the x level and three copies at the 1s level. The answer must be $2x + 3$. But what if we were in a $1 \leftarrow 10$ machine and computed $276 \div 12$? We'd get the same exact picture! What could be considered an advanced high school algebra problem is actually just a repeat of an early grade-school arithmetic problem! [KB]

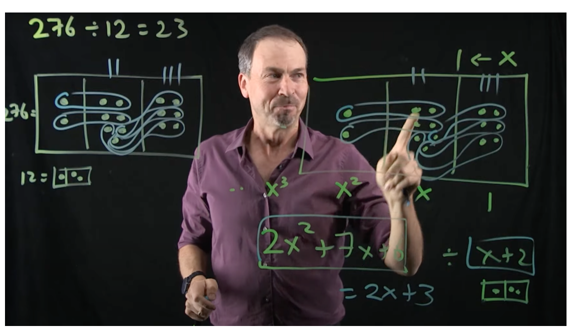


Figure II.11: Same Picture

What if we had the example $\frac{x^3-3x+2}{x+2}$ that doesn't work out so nicely since we have to deal with negative numbers? Well we just represent $x^3 - 3x + 2$ as $|1||0||-3||2|$ in an $1 \leftarrow x$

machine. And we are looking for copies of $x+2$, that is one dot next to 2 dots, anywhere in the picture of x^3-3x+2 and we don't see any. And we can't unexplode dots because we don't know how many dots to draw since we don't know the value of x. However, a great piece of life advice is: "If there's something in life you want, make it happen! But then be sure to deal with the consequences." This is an amazing quote by James Tanton.

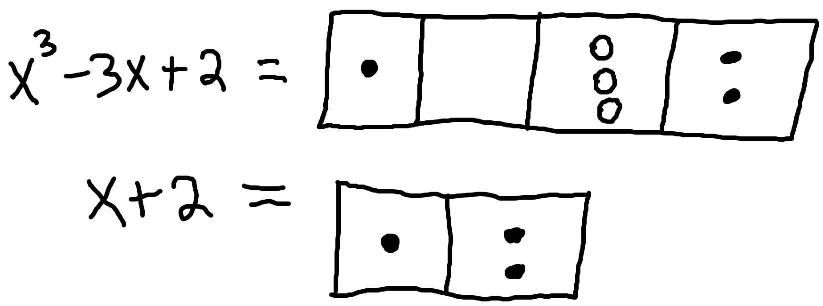


Figure II.12: Pattern isn't there

Looking at this picture we have a single dot way down at the left and it would sure be nice to have two dots in the box next to it to make a copy of $x+2$. So we can just go ahead and put in two dots but there are consequences if we do that. Since that box is meant to be empty we can just put in two antidots as well. Then we have two dots in our 1s box and it would be nice to have a dot in the x box to go with it. So we put in another dot and antidot pair and we find another copy of $x+2$. But now look at the following picture II.13 and see if you notice anything.

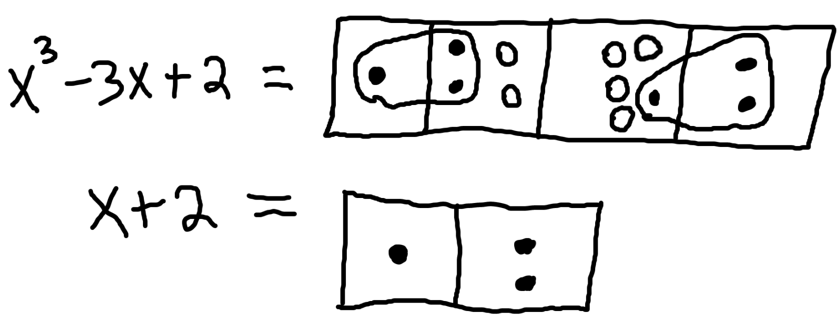


Figure II.13: Exact opposite of what we want

Instead of one dot next to two dots we see copies of the exact opposite of what we're looking for. We have two copies of one antidot next to two antdots. So we see that $\frac{x^3-3x+2}{x+2} = x^2-2x+1$.

Jumping back to base ten, let's consider $31,824$ divided by 102. Think how unpleasant that would be using the traditional algorithm. It is a breeze with Exploding Dots! 102 looks like one dot, zero dots, two dots. So we just look for that pattern anywhere in the picture of $31,824$. We keep in mind that all of the dots must really reside in the rightmost part of the loop. The solution of 312 instantly appears. You're encouraged to watch James do this example on his YouTube channel. Can we now go back to polynomial division and handle

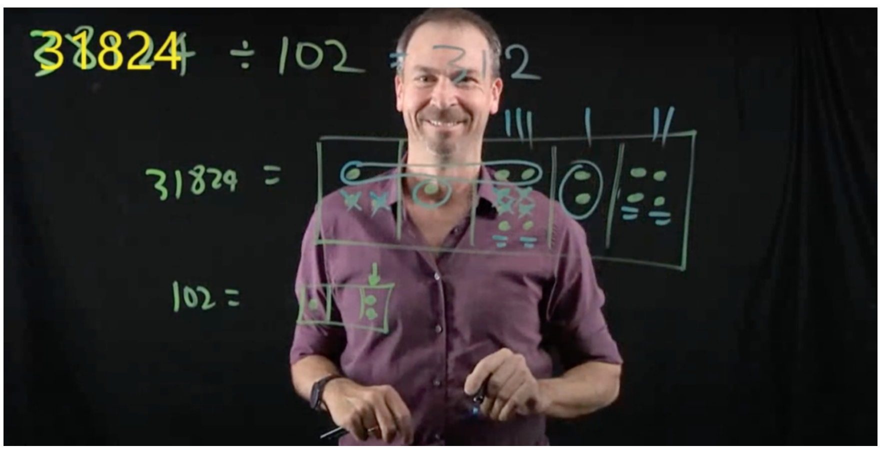


Figure II.14: $31,824 \div 102$

this next one?

Consider $\frac{4x^5-2x^4+7x^3-4x^2+6x-1}{x^2-x+1}$. That looks nasty! But using Exploding Dots it's just as simple to compute and actually quite a bit of fun. Take a look at our numerator represented in II.15.

So the pattern we want is one dot next to one antidot next to one dot. Loops here could get very messy so it might be better to just circle each dot or antidot individually and then make a tally mark to keep track of which level the group resides in. Once we account for all copies of x^2-x+1 we can create more by again just inserting dot and antidot pairs. We then are looking at the following picture in II.16.

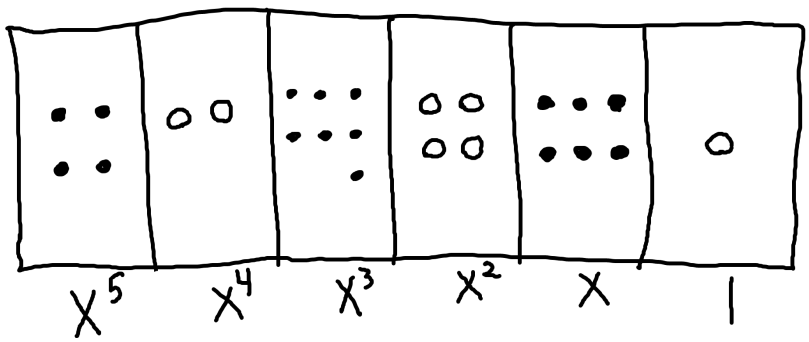


Figure II.15: Here's the numerator

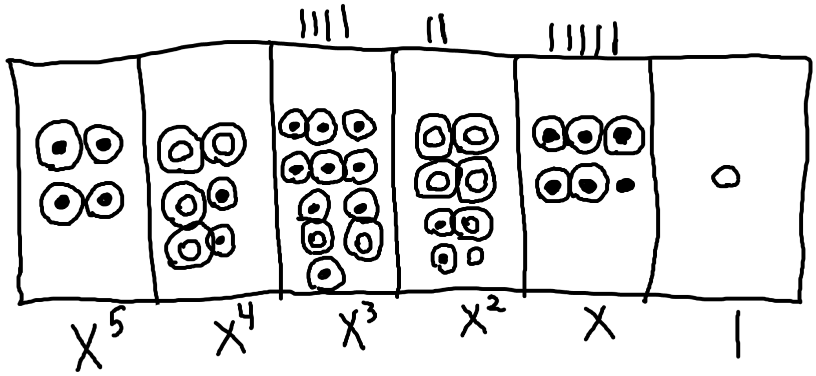


Figure II.16: We have an anti-pattern

So we want a dot next to an antidot next to a dot, but look what we have! We have an antidot next to a dot next to an antidot, which is the exact opposite of what we're looking for. We therefore have an anti version at the 1s level. And thus we see that $\frac{4x^5-2x^4+7x^3-4x^2+6x-1}{x^2-x+1} = 4x^3+2x^2+5x-1$. Pretty cool, but wait a minute. What if x really was equal to ten? Well plug in ten and you'll see that we just computed $\frac{386{,}659}{91} = 4,249$. But of course x can be any value in the domain. Whatever value we substitute in for x puts us in that base. We just computed an infinitude of division problems all at once!

In all of the machines that we have discussed so far we have had a rightmost box with boxes then extending infinitely far to the left. But that does seem to be awfully lopsided. Could we have boxes that extend off to the right as far as we desire too? Well of course we can! If we were in a $1 \leftarrow 2$ machine then we'd still have the rule that two dots in a box would become one dot one space to the left. Therefore each dot in the box to the right of the 1s box must equal $\frac{1}{2}$ since two $\frac{1}{2}s$ is 1. And two $\frac{1}{4}s$ is $\frac{1}{2}$. So we have our boxes to the right as the reciprocal powers of 2. We would typically use a dot, or we should probably say a point, to separate the two sides of the machine. So we have just discovered decimals, or maybe we should call them bimals here since we're in binary. But we'd see that we'd have the reciprocal powers of b in any base b for our boxes extending to the right of our 1s box. In II.17 we see a base 3 example for finding the "decimal" representation of $\frac{1}{4}$ in a $1 \leftarrow 3$ machine. Well 4 in base 3 is 11, so we're looking for one dot next to one dot. We start with one dot in our 1s box which we can unexplode to three dots in the next box to the right. We can then unexplode one of those dots to see two groups at our $\frac{1}{3^2}$ or $\frac{1}{9}$ level.

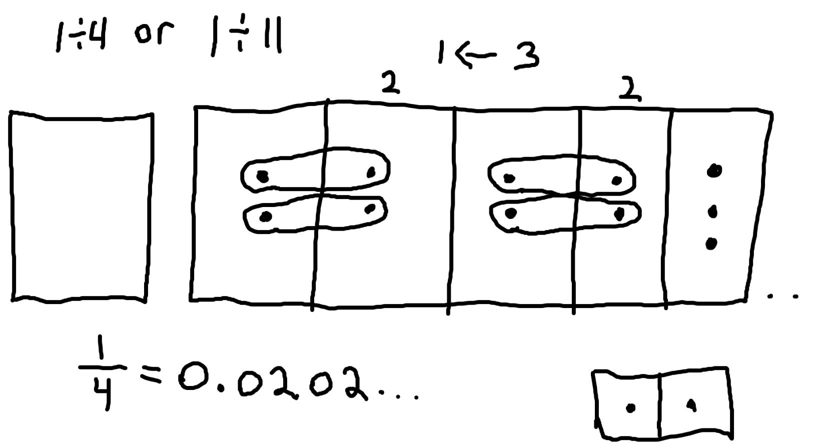


Figure II.17: $1 \div 4$ in base 3

We can then keep repeating this process and see that we'll be doing this forever. Therefore we see that $\frac{1}{4}$ or one quarter in base 3 is the repeating "decimal" 0.020202.... Let's go back to base ten for a bit and see how Exploding Dots can explain and prove some divisibility theorems.

II.1 Divisibility by 9

The following is a well-known theorem.

Theorem 1.1. *A number is divisible by 9 $\iff$ the sum of its digits is divisible by 9. In fact, a number leaves the same remainder upon division by 9 as does the sum of its digits.*

For example, we can say that 25821 is divisible by 9 since $2+5+8+2+1 = 18$ is divisible by 9. And 40061 is two more than a multiple of 9 ($40061 = 4451 \times 9 + 2$) and $4+0+0+6+1$ is also two more than a multiple of 9.

Proof. Each dot in a $1 \leftarrow 10$ machine leaves a remainder of 1 upon division by 9. This following image II.18 illustrates why as we can unexplode any dot in our machine to find one group of 9 with one dot left over. Therefore we have: If a number is represented by n

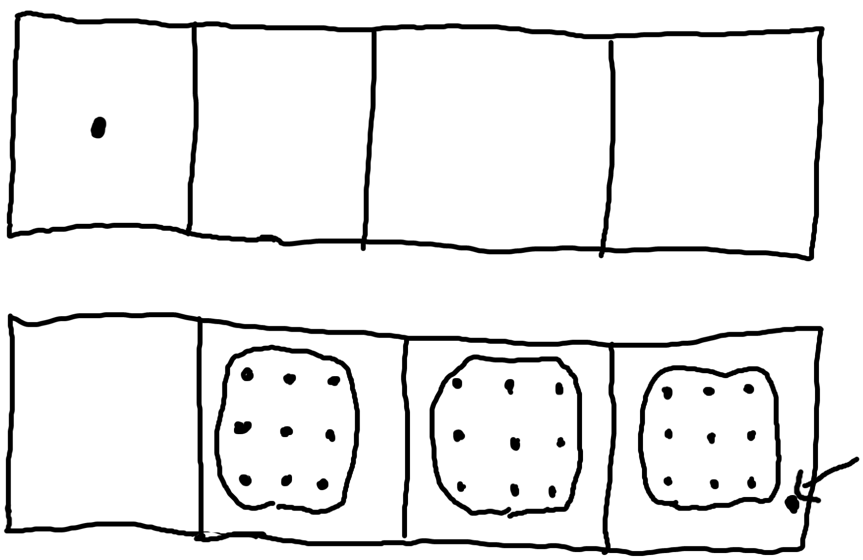


Figure II.18: Any dots in the rightmost box are precisely the dots we started with

total dots in a $1 \leftarrow 10$ machine, dividing by 9 leaves us with n dots in the rightmost box. And n is the sum of the digits in our original dividend. □

II.2 Arithmetic "trick" explained with Exploding Dots

Here is an interesting and unusual way to divide by 9. Let's say we wanted to compute $23102 \div 9$. What we can do is just read the dividend but as we go along take the partial sum. So we have $2+3$, then $2+3+1$, and so on, until our last sum is our remainder. So we can write $23102 \div 9 = 2566R8$. Exploding Dots shows why this works as we think of 9 as $10-1$ in our machine as illustrated in II.19.

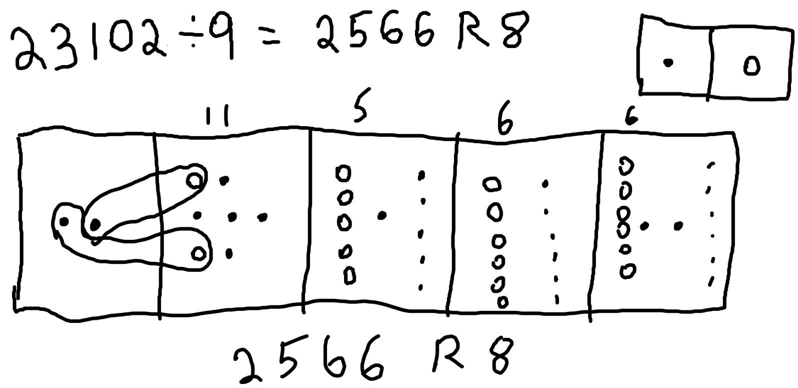


Figure II.19: 2566 R8

II.3 Divisibility by 7

In [SA] the author begins with a chapter on the basics of number theory and details several divisibility rules along with their proofs. However it is stated that *the test for divisibility by 7 is complicated and not used much, so we omit it.* The test does seem to be complicated as well as mysterious but Exploding Dots clearly illustrates why this rule will always work. The rule states that if we have a given base ten number that is divisible by 7 we can delete the final or $1s$ digit and then subtract 2 times that deleted digit from the new number that remains after we delete the end digit. Our original number is divisible by 7 iff the new number we obtain is. We can then just repeat this process until we obtain a number that we know is divisible by 7 or not. That does seem weird and mysterious so let's consider an example, $39872 \div 7$. We can compute the following as shown in II.20: So if we repeatedly delete the final digit and then subtract two times this digit from the number that remains we come to 35 which is easily seen to be divisible by 7. Then according to this rule the original number, in this case 39872 must be divisible by 7 too. This works because if we

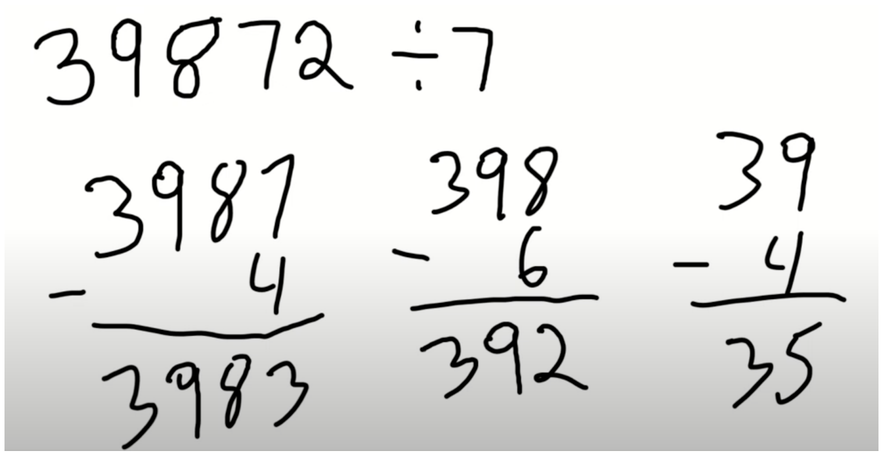


Figure II.20: mysterious rule

start with any number that is divisible by 7, then adding or subtracting multiples of 7 will not affect if the number is divisible by 7. So if we represented some number in a $1 \leftarrow 10$ machine deleting the final digit would mean we would be inserting as many antidots as we have dots in the rightmost box. Let's first assume we have just a single dot in the $1s$ place. If we insert an antidot in the rightmost box then we could insert two antidots in the next box to the left. This is because two antidots next to one antidot is the anti version of 21. So in other words we are subtracting 21. But 21 is a multiple of 7 so we can subtract any lots of 21 and if the original number was divisible by 7 then the new number must be too. This clearly shows why this divisibility rule for 7 works. This mysterious divisibility rule for

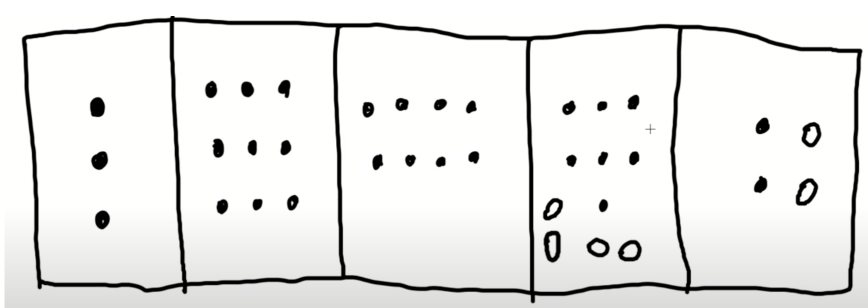


Figure II.21: simple proof for divisibility by 7 rule

7 becomes extremely straightforward when viewed with Exploding Dots.

II.4 Just as easy to identify remainders dividing in base x as in base ten

How would we compute $\frac{x^4}{x^2-3}$? Just to clarify, we don't mean division in the ring of rational expressions. When we write $\frac{x^4}{x^2-3}$ we mean $x^4 \div (x^2 - 3)$. Here in II.22 is our picture of x^4 in our $1 \leftarrow x$ machine and $x^2 - 3$ looks like one dot, then zero dots, then three antidots. We clearly don't have that pattern to begin with, so let's just create what we want and deal

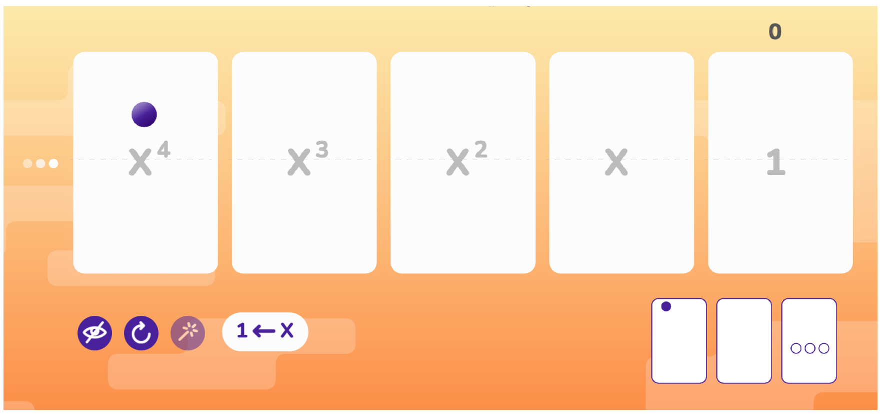


Figure II.22: $\frac{x^4}{x^2-3}$

with the consequences of doing so. We now see that we have a pattern at the x^2 level but

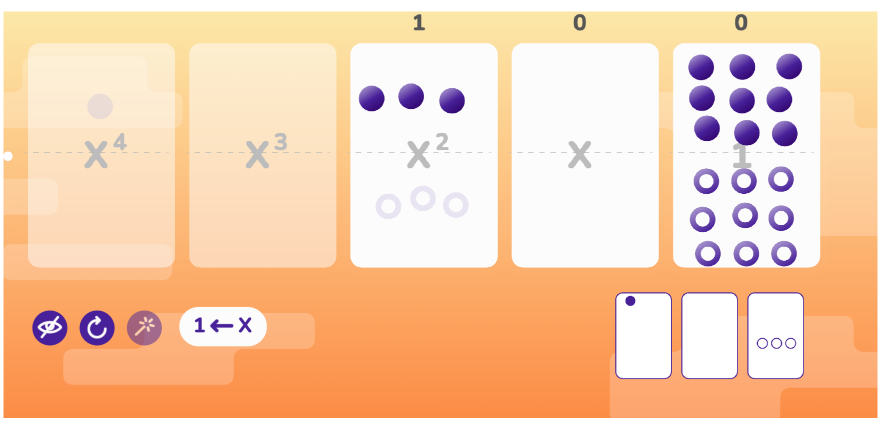


Figure II.23: We have a remainder of 9

have accumulated 3 more dots that we'd like to have 3 antidots two boxes down to go with them.

So we then are looking at the following picture in II.23. And we have three patterns at the $1s$ level but have nine dots left over, which means that we have 9 dots that are still waiting to be divided by $x^2 - 3$. This means that $\frac{x^4}{x^2-3} = x^2 + 3 + \frac{9}{x^2-3}$. This is the same as saying $\frac{x^4}{x^2-3} = x^2 + 3$ R 9 writing our quotient with remainder notation.

II.5 Discover the Fibonacci sequence with Exploding Dots

In the appendices we will explore what happens when we evaluate $\frac{1}{1-x}$ which leads us to an infinite sum. Here we are going to be computing $\frac{1}{1-x-x^2}$, or in other words $1 \div (1 - x - x^2)$. Our claim is that if we compute $\frac{1}{1-x-x^2}$ we will obtain $1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + ... = \sum_{n=0}^{\infty} F_n x^n$ where F_n is the *nth* Fibonacci number starting with $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$.

Proof. Consider just one dot in an $1 \leftarrow x$ machine. Since we are dividing $\frac{1}{1-x-x^2}$ we want to see how many groups of one antidot next to one antidot next to one dot we can find. We obviously don't have this pattern to start with, so let's create the pattern we want and then deal with the consequences of doing so.

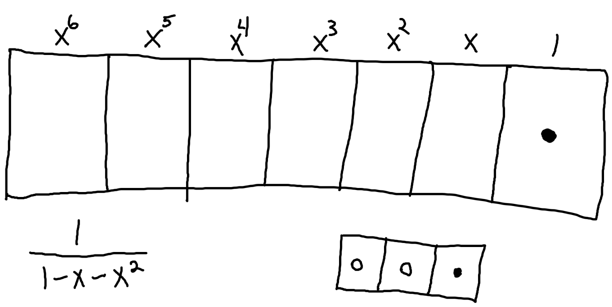


Figure II.24: $\frac{1}{1-x-x^2}$

We can see in II.25 that we've got a pattern at the $1s$ level, but to create our pattern we needed to enter in a dot to each box that we entered an antidot. Since we currently have one dot in our x box we can create one pattern at the x level. But now we picked up an extra dot

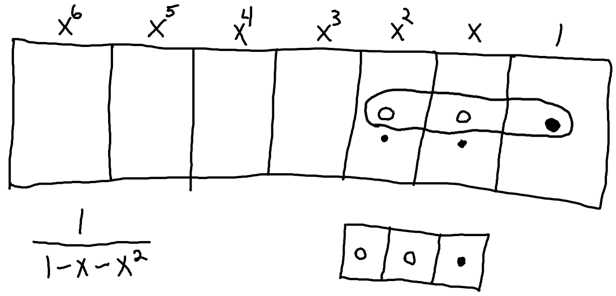


Figure II.25: $\frac{1}{1-x-x^2}$

at the x^2 level so we can create two patterns at that level. Then we have three dots in our x^3 box, so to create another pattern we'd put in three antidots in the next two boxes which would add three more dots to the two dots already in the x^4 box. Continuing we realize we'd be doing this forever to give us an infinite sum where the number of solid dots in each box give us the Fibonacci numbers. So therefore $\frac{1}{1-x-x^2} = 1+x+2x^2+3x^3+5x^4+8x^5+13x^6+...$ where the coefficients on each power of x give us the sequence of Fibonacci numbers.

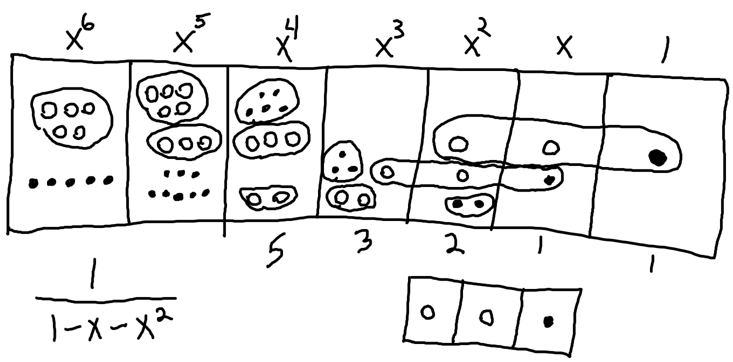


Figure II.26: $\frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n$

□

.

The Division Algorithm and Euclidean Domains

We are likely familiar with the Division Algorithm, nonetheless we will define this theorem for both $\mathbb{N}$ and $\mathbb{Z}$.

Theorem 5.1. *(Division Algorithm for $\mathbb{N}$) Suppose a and b are natural numbers and that $b \leq a$. Then there is a natural number q and positive integer r such that $a = bq + r$ and $0 \leq r < b$. Furthermore, q and r are unique.*

(We usually call q the "quotient" and r the "remainder" when a is divided by b.)

We can also state a division algorithm for $\mathbb{Z}$.

Theorem 5.2. *(Division Algorithm for $\mathbb{Z}$) Suppose $a, b \in \mathbb{Z}$ and $b > 0$. Then there exist unique $q, r \in \mathbb{Z}$ such that $a = qb + r$ and $0 \leq r < b$, where q is the quotient and r is the remainder.*

This is the usual "long division" familiar to us from elementary arithmetic.

Here is an example in $\mathbb{N}$:

Example 5.3. *If we have $7 \div 3$, $3 < 7$, so we can write $7 = 3q + r$ where $0 \leq r < 2$ Namely, we have $q = 2$ and $r = 1$.*

And here is an example in $\mathbb{Z}$ involving negative numbers:

Example 5.4. *Let's say we have $b = 3$ and $a = -7$. We can write $-7 = 3q + r$, where $0 \leq r < 3$. Here the values that work are $q = -3$ and $r = 2$, and this is the only way to* *pick q and r so that $0 \leq r < 3$. This following image III.27 illustrates how we begin with seven antidots and are looking for groups of three. Two groups are immediately apparent and since we want a positive remainder we put in two dot/antidot pairs to then see one more anti-version of what we're looking for.*

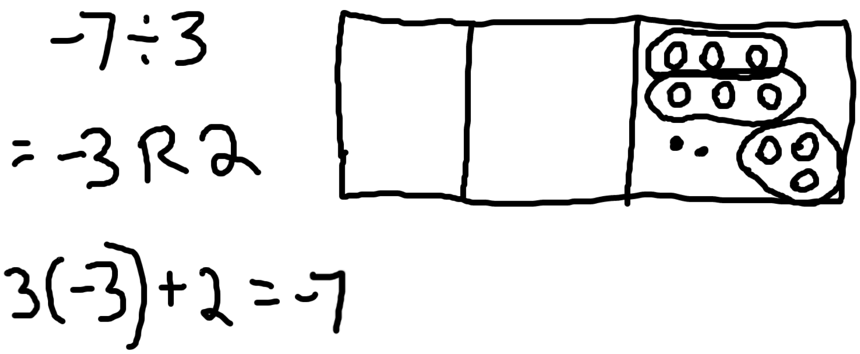


Figure III.27: Division algorithm in $\mathbb{Z}$

The division algorithm allows us to take two given integers n and d and compute their quotient q and remainder r where $0 \leq r < |d|$. Similar to how multiplication can be thought of as repeated addition, division can be thought of as repeated subtraction. If we divide our dividend n by our divisor d we will subtract d from n repeatedly, i.e. $n - d - d - d - d - ...$ until we get a result that lies between 0 (inclusive) and d (exclusive) and is the smallest non-negative number obtained by repeated subtraction. Then the resulting number is known as the remainder r, and the number of times that d is subtracted is called the quotient q.

III.1 Euclidean Domains

We will define the notion of a *norm* on an integral domain R. This is essentially giving a measure of "size" in R. We will then define a *Euclidean Domain.* Recall that an integral domain is a commutative ring R with multiplicative identity 1 and no zero divisors. Or we can say that a non-zero commutative ring is an integral domain if and only if $\forall a, b \neq 0 \implies$

$ab \neq 0$. The ring of integers and the ring of polynomials are certainly examples of integral domains.

Definition 1.1. *Any function $N : R \to \mathbb{Z}^+ \cup 0$ with $N(0) = 0$ is called a* norm *on the integral domain R. If $N(a) > 0$ for $a \neq 0$ define N to be a* positive norm. *[DF]*

Definition 1.2. *An integral domain R is said to be a* <u>*Euclidean Domain*</u> *(or possess a division algorithm) if there is a norm N on R such that for any two elements a and b of R with $b \neq 0$ there exist elements q and r in R with*

$$a = qb + r \text{ with } r = 0 \text{ or } N(r) < N(b).$$

The element q here is called the quotient and the element r is the remainder of the division.

III.2 Examples of Euclidean Domains

Example 2.1. *The integers $\mathbb{Z}$ are a Euclidean Domain with norm given by $N(a) = |a|$, the usual absolute value. [DF]*

Example 2.2. *If F is a field, then the polynomial ring $F[x]$ is a Euclidean Domain with norm given by $N(p(x))$ = the degree of $p(x)$. The division algorithm for polynomials is simply "long division" of polynomials.*

So if we have a polynomial ring $F[x]$ that is a Euclidean Domain then if $a(x)$ and $b(x)$ are two polynomials in $F[x]$ with $b(x)$ nonzero, then there are unique $q(x)$ and $r(x)$ in $F(x)$ such that $a(x) = q(x)b(x) + r(x)$ with $r(x) = 0$ or the degree $r(x) <$ degree $b(x)$.

The Gaussian integers are another interesting example of a Euclidean Domain that we will explore in the next chapter. The ring of integers, the ring of polynomials, and the ring of Gaussian integers $\mathbb{Z}[i]$ are all examples of Euclidean Domains. We can therefore perform division in these Euclidean domains and Exploding Dots allows one to feel empowered by beautifully elucidating and conceptualizing these ideas.

Gaussian Integers

Gaussian integers are complex numbers whose real and imaginary parts are both integers. The Gaussian integers, with ordinary addition and multiplication of complex numbers, form the integral domain $\mathbb{Z}[i]$. Formally, Gaussian integers are the set $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$.

Here are some examples of Gaussian and non-Gaussian integers. $2 + 3i$, $4 + 7i$, 17, and 0 are all Gaussian integers, while $\frac{4}{3}$, $\sqrt{2}$, and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ are not.

The norm of a Gaussian integer is the square of its absolute value, as a complex number. It is the positive integer defined as $N(a + bi) = (a + bi)(a - bi) = a^2 + b^2$. The reason we prefer to deal with norms on $\mathbb{Z}[i]$ instead of absolute values on $\mathbb{Z}[i]$ is that norms are integers (rather than square roots), and the divisibility properties of norms in $\mathbb{Z}$ will provide important information about divisibility properties in $\mathbb{Z}[i]$. [KC]

A Euclidean division algorithm takes, in the ring of Gaussian integers, a dividend a and divisor b $\neq$ 0 and produces a quotient q and remainder r such that $a = bq + r$ and $N(r) < N(b)$. In fact, we can actually do better than this, one can make the remainder smaller to obtain $a = bq + r$ and $N(r) \leq \frac{N(b)}{2}$. It is well-known that the Gaussian integers are a Euclidean domain. In this chapter we will show this using Exploding Dots. We will show as we perform the division that one can make the remainder smaller. Now for us to be able to divide Gaussian integers with Exploding Dots we will need to use our imagination to envision what a base i machine would look like.

IV.1 A base i machine

Exploding Dots can show how the same concept we have used to solve basic arithmetic and algebra problems can also be used to solve problems involving complex numbers, in particular Gaussian integers. To do so we will be using a base i machine [KB]. But what would this base i machine look like? Well, it would seem like we can't really develop a rule for an $1 \leftarrow i$ machine, but we know what we want the values of the boxes to be. The same as any $1 \leftarrow n$ machine the boxes would begin from the right and head off to the left with values $i^0, i^1, i^2, i^3, i^4, i^5, i^6, i^7$, and so on. But we know that by the properties of the imaginary number i that either

$$i^{2n} = 1 \qquad i^{2n+1} = i$$

$$i^{2n} = -1 \qquad i^{2n+1} = -i$$

So if we apply these properties we notice that our rightmost box is 1, then our next box is i, then -1, then $-i$ and these values repeat every four boxes. So going from right to left we have the cyclic pattern of values $(1, i, -1, -i)$ and this keeps repeating.

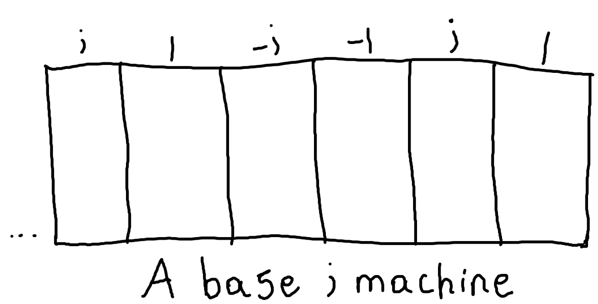


Figure IV.28: Exploding Dots machine with repeating box values

Hence we can make this base i Exploding Dots machine circular. A property of this Circular Exploding Dots machine is that a dot in the box labeled 1 can always be moved to an antidot in the box labeled -1 and vice-versa. Similarly, a dot in the box labeled i can always be moved to an antidot in the box labeled $-i$ and vice-versa. So we could use antidots if we'd like, but we should be able to represent any Gaussian integer in this machine with only dots. Recall that a Gaussian integer is just a complex number of the form $a + bi$ where a and b are both integers. The following figure IV.29 is a visual puzzle to determine the odd one out. You're encouraged to look at the six circular Exploding Dot machines in IV.29 to determine what Gaussian integer is represented in each machine before reading on. We can easily see that all the machines represent $3 + 2i$ except the last one which represents $1 + 2i$. Now that we understand how this machine works, we're all set to solve problems involving Gaussian integers.

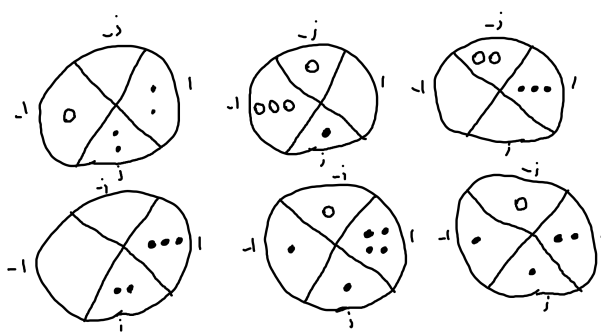


Figure IV.29: Find the odd one out

IV.2 Division with Circular Exploding Dots

The material in this section has been mentioned in the GMP newsletter [JT].

The way we're labeled here in IV.30, $1 + 2i$ looks like one dot next to two dots going in the counter-clockwise direction. So here we're dividing 5 by $1 + 2i$ and we start with 5 dots in our 1s box. So we want groups of $1 + 2i$ which looks like 1 dot next to two dots going in the counter-clockwise direction. We don't see any to start with so we can just add two dots in the i box as long as we add two dots in the $-i$ box too. That's the same as adding two dots and two anti-dots which we know we can always do because they'd annihilate each other. We then would see one group of the pattern we're looking for at the 1s level and two

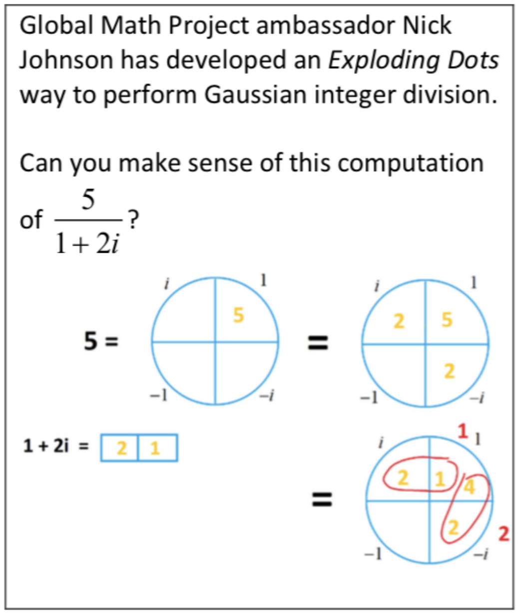


Figure IV.30: $\frac{5}{1+2i} = 1 - 2i$

groups at the $-i$ level.

Here's an example in IV.31 of $\frac{3+7i}{4+2i}$.

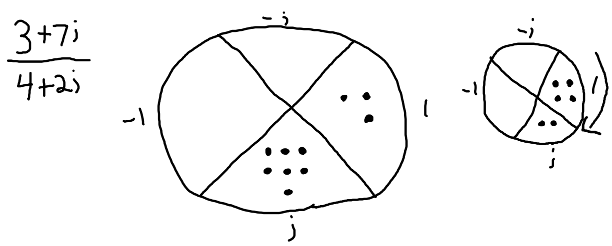


Figure IV.31: $\frac{3+7i}{4+2i}$

So we have $3+7i$ represented in our base i machine and we want to find groups of $4+2i$ which looks like 4 dots next to 2 dots going in the clockwise direction. We can see from this next image IV.32 that we get a quotient of $1+i$ with a remainder of $1+i$ and this can be easily verified as just as in normal Euclidean division when we divide a by b we get a quotient q and remainder r such that $a = bq + r$.

Let's now look at IV.32 where we are dividing the Gaussian integers $\frac{7}{1-2i}$ in a circular

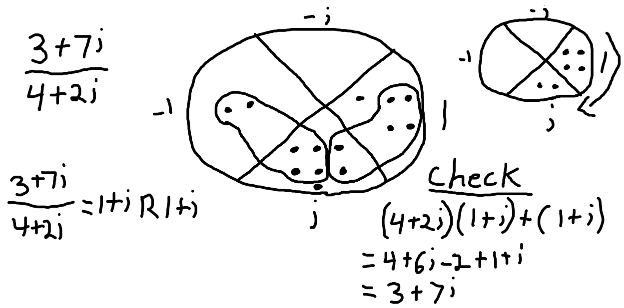


Figure IV.32: $(4+2i)(1+i)+(1+i) = 3+7i$

base i Exploding Dots machine.

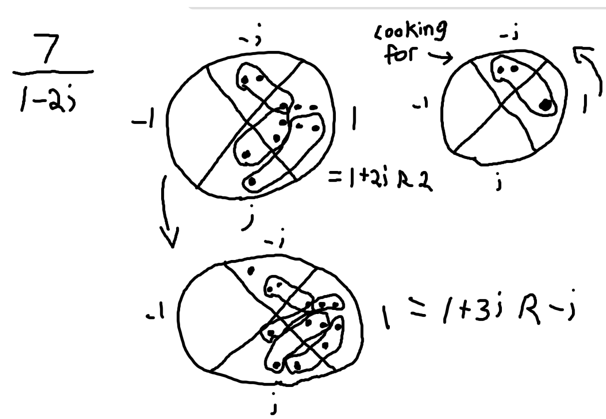


Figure IV.33: $\frac{7}{1-2i}$

To create the pattern we want we insert dots into our i and $-i$ cells and we have one group at the $1s$ level and two groups at the i level with two dots left over in the $1s$ cell. Hence our quotient is $1+2i$ with a remainder of 2.

And we do have the norm of the remainder less than the norm of the divisor. But wait! It is painstakingly obvious from the picture that we can go further! We could easily put in a dot in our i and $-i$ cells to create another pattern at the i level to then only leave one remainder dot in the $-i$ cell. So we see in IV.33 the norm has then been reduced from 4 to 1.

This next picture IV.34 is a visual that shows when we divide two Gaussian integers that the norm of the remainder can always be made less than half of the norm of the divisor. The circle is meant to have radius $\frac{|b|}{\sqrt{2}}$.

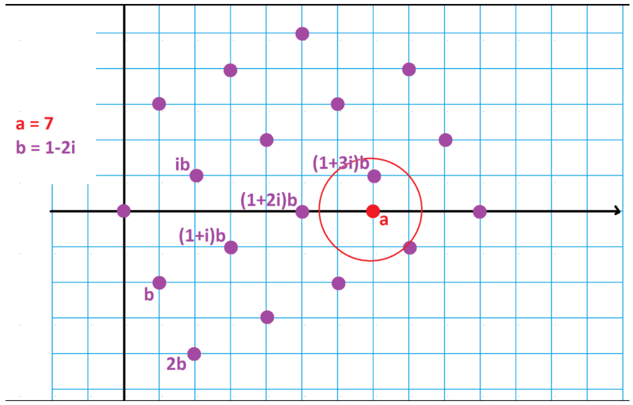


Figure IV.34: Gaussian integers in the complex plane

Theorem 2.1. *For any two Gaussian integers a and b, there is a multiple of b, call it qb, at distance at most* $\frac{|b|}{\sqrt{2}}$ *from a.*

We want to show that given a,b $\in \mathbb{Z}[i], \exists q \in \mathbb{Z}[i]$ such that $N(a-bq) \leq \frac{N(b)}{2}$

Proof. Let a and b be any two Gaussian integers. Write $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$ with a_1, a_2, b_1, b_2 being integers.

All the multiples of b make a square lattice of Gaussian integers. These multiples of b are the purple dots in the picture. The tilted squares have side length $|b| = \sqrt{b_1^2 + b_2^2)} = \sqrt{N(b)}$. We can see this by the Pythagorean Theorem.

The red point a sits in one of those squares.

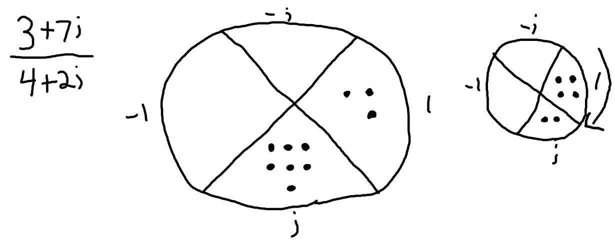
Now, any point in a square of side length $|b|$ is at most $\frac{|b|}{\sqrt{2}}$ away from a corner of that square. That is because $\frac{1}{2}$ of the diagonal of the square is equal to $\frac{|b|}{\sqrt{2}}$ (The center point is the furthest from a corner.) So regardless of where the point a lies in the square, it will always be within $\frac{|b|}{\sqrt{2}}$ from a corner and thus a multiple of b.

□

Corollary 2.2. *$a = qb + r$ with r a Gaussian integer satisfying* $|r| \leq \frac{|b|}{\sqrt{2}}$

Proof. Set $r = a - qb$. □

Now if we are dividing two Gaussian integers $a \div b$ where our divisor is b and our current dividend is a, we would like to subtract one of the four possibilities $b, ib, -b$, or $-ib$ in order to minimize the norm of the difference. In the picture of the complex plane, we want to make the move that takes us closest to the origin. From the algebra of dot or inner products we have for any number c, $||a-c||^2 = ||a||^2 - 2 < a, c > + ||c||^2$ so we will minimize the new distance to the origin by choosing c to be whichever one of $b, ib, -b$ or $-ib$ maximizes the inner product $< a, c >$. This gives us an arithmetic way of looking at the dots in our circular Exploding Dots machine. Looking at IV.31 again, we can see that there must be four ways



to remove the pattern we're looking for, although some of them may involve adding a lot of dot/antidot pairs. Some of these options we can see are obviously wrong but they still must be there. When considering the dot product idea, the obviously wrong options would correspond to choosing a value for c which would move us further away from the origin. So generically, two of the options for c should move us closer to the origin, and the opposite two would move us further away.

Claim: If $N(a) > \frac{1}{2}N(b)$, $i.e. |a| > \frac{1}{\sqrt{2}}|b|$, then there is "move" which decreases $N(a)$. Here by a move we mean to subtract one of the four possible values $b, ib, -b, -ib$. These moves are 90° apart so there must be one that is no more than 45° from the origin. Before we prove the claim, let's first consider the case where we have equality. That is $|a| = \frac{|b|}{\sqrt{2}}$.

We will draw a circle centered at the origin O in IV.35, with radius $|a|$. Call this circle with radius $\frac{|b|}{\sqrt{2}}$ the critical circle. In this case we draw rays making an angle of 45° on either side of Oa. It is enough to look at just one of these rays since the situation on the other side of Oa is symmetric. We let a' be the intersection of this ray with the circle.

$Oa = \frac{1}{\sqrt{2}}|b| = Oa'$, so we have that $\triangle aOa'$ is isosceles. And the $m(\angle Oa'a) = 45°$, which means that the $m(\angle aOa') = 90°$.

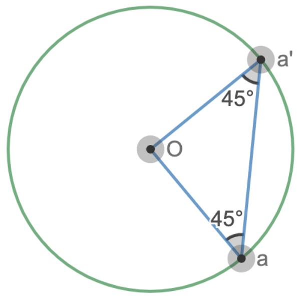


Figure IV.35: circle centered at O with radius $\frac{|b|}{\sqrt{2}}$

Because we have a right triangle, the Pythagorean Theorem tells us that the distance form a to a' is $\sqrt{2}$ times the radius of the circle, *i.e.*$\sqrt{2}|a|$ which is equal to $|b|$. In other words, a move of length $|b|$ at an angle of $45°$ from segment Oa (which is the furthest from the origin that we can ever be forced to move) will leave us exactly the same distance from the origin as our starting point.

We now deal with the case stated in the claim, where $N(a) > \frac{1}{2}N(b)$, so in this case a is outside the critical circle. We are saying that if a is outside the critical circle there will be a move that moves closer to the origin and within finitely many steps we will be within the critical circle.

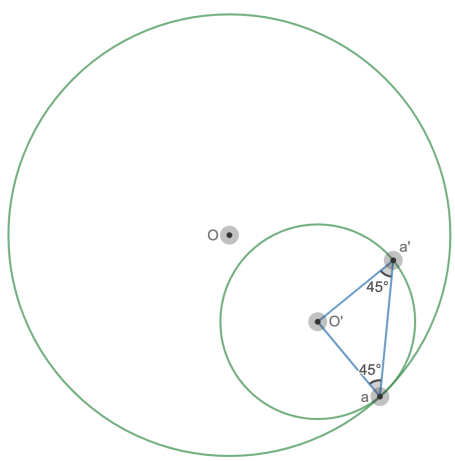


Figure IV.36: If $N(a) > \frac{1}{2}N(b)$ we can move closer to the origin

We see in IV.36 that we translate the critical circle along the segment Oa until its circumference meets a. The translated circle now lies inside the circle centered at O with radius $|a|$.

We let O' be the the center of the translated circle. By the first case we see that if we move $|b|$, even at the worst possible $45°$, we will move onto the critical circle. So no matter what we will be inside the circle of radius $|a|$. In all cases we therefore will move inside the circle of radius $|a|$. In other words, we will decrease the norm of a, proving the claim.

So if $N(a) > \frac{1}{2}N(b)$ we can always decrease the norm of a. In other words, a greedy algorithm will always find the quotient and remainder. Note that $N(a)$ is an integer so it will be decreased by at least 1. We now see that we can find the quotient and remainder in the division algorithm for Gaussian integers by using a greedy algorithm. At each stage we choose the move which results in the smallest norm of a. As long as $N(a) > \frac{1}{2}N(b)$ we can choose a move and keep subtracting.

This can be done visually and intuitively in the base i machine. Here we have an example of dividing $\frac{9}{2+i}$. In IV.37 we are first subtracting b which gives us a quotient of 1 with a remainder of $7 - i$ and then we're subtracting ib which gives us a quotient of 0 with a remainder of 9. So in the second case we don't get anywhere.

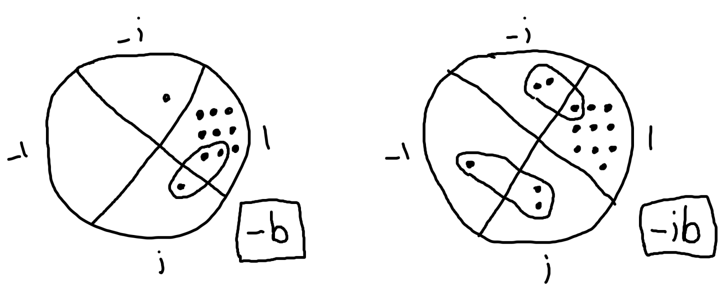


Figure IV.37: $\frac{9}{2+i}$

Then in IV.38 we are first subtracting $-b$ which again gives us a quotient of 0 with a remainder of 9. And then we are subtracting $-ib$ which gives us a quotient of $1 - i$ with a remainder of $6 + i$.

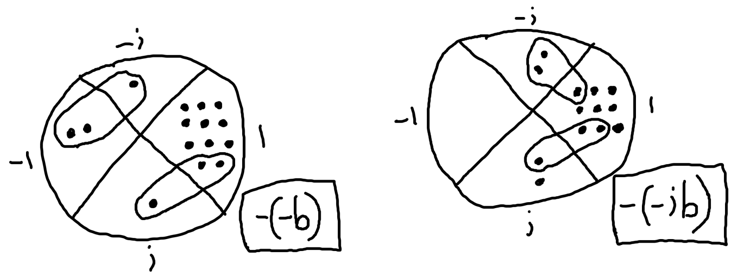


Figure IV.38: $\frac{9}{2+i}$

Or this is how you might naturally approach it as we see in IV.39. So if we're dividing $\frac{9}{2+i}$ we start with 9 dots in our $1s$ cell and we're looking for groups of $2+i$ which looks like 2 dots next to 1 dot going in the clockwise direction. Well in eight of those nine dots we have 4 groups of 2. So we want to have 4 dots in our i box to go with them. But if we put in 4 dots in the i box, then we need to put in 4 dots in the $-i$ box to counteract them. We then have 4 groups at the $1s$ level and 1 group at the $-i$ level with two dots left over in the $-i$ box. We thus see we have a quotient of $4-i$ with a remainder of $-2i$. We do note that the norm of the remainder is less than the norm of the divisor which satisfies the requirement for a Euclidean Domain. But again it is so clear from the picture that we can go further! We could easily put in a dot in the $1s$ box to give us another pattern, but then of course we need to put in a dot in the -1 box too. We then get a quotient of $4-2i$ with a remainder of -1.

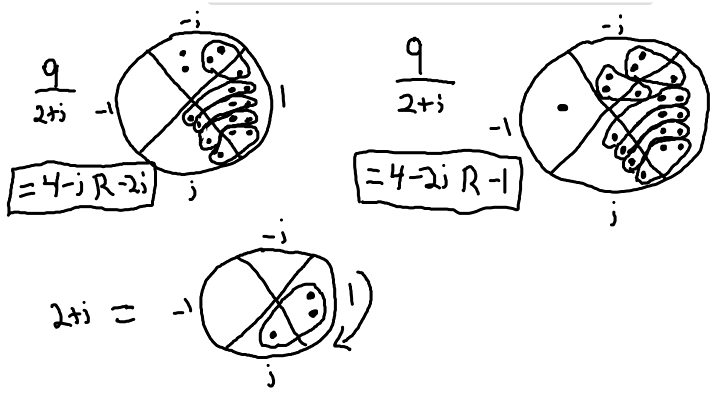


Figure IV.39: $\frac{9}{2+i}$

Conclusion

We have seen how the Exploding Dots approach to representing place value and arithmetic is a very simple idea indeed. The Global Math Project launched officially at MoMath, the Museum of Mathematics, in the fall of 2017. Its hope was to enthrall one million people around the world in its inaugural year with a piece of joyous mathematics called *Exploding Dots*. This was marketed as a mind-blowing experience, created by Dr. James Tanton, that uses the concrete idea of dots and some basic arithmetic sense to propel people from simple binary expressions to calculus and beyond to unsolved research problems. This, possibly seen as quirky, dots-and-boxes approach can and should be utilized in all classrooms ranging from early elementary school right on through to the college level. When students are able to play and get to experience the feeling of true joyous understanding, genuine learning will certainly follow, and their curiosity will pull them down the rabbit hole of mathematical awe and wonder.

One of the most magical books ever written about the joy of mathematics is *Pi of Life: The Hidden Happiness of Mathematics* by another ambassador for *The Global Math Project* Sunil Singh. James Tanton writes the foreword in Sunil's lovely book which begins like so: [SS]

"What is mathematics?

Math is humility, simplicity, courage, curiosity, gratitude, health, power, resilience, laughter, connection, and hope. Math is that which speaks to human truth and soulful joy, and yet transcends our humanness. It is the paradox of creativity and utility united. It is the surprise of discovering something you felt you somehow knew all along, but didn't. To stare into math is to stare into the cosmos, to experience visceral alarm and ebullient joy hand in hand, to see one's own insignificance and yet find meaning simply from being. Mathematics is the portal to the playground of the soul."

Exploding Dots may, or likely may not, be awe-inspiring to a mathematician. It likely might not do much for a mathematician because it represents on a page what a mathematician has in their mind. But this is why it can be so effective for a learner to arrive at this deep level of understanding. This might sound reminiscent of some of the initial praise of the "New Math" of the 1960s which ended up to be quite a disaster after it was implemented. There's even a famous song parody by Tom Lehrer about this New Math, which you've got to find quite funny regardless of what side you take in the present so-called "math wars." There are certainly some similarities between Exploding Dots and the New Math of the 60s but Exploding Dots really is so much different. Nobody gets "Exploding Dots anxiety" because it begins as a story which doesn't look like math. It's just playing and having fun with these machines and finding secret codes for numbers. Once students get to the $1 \leftarrow 10$ machine they get to have that mind-blowing realization that they can write numbers in any base but also realize that so many of the algorithmic processes they've learned in school, using of course base ten, is just Exploding Dots without the dots. And there is the big conceptual link between writing integers in multiple bases and working with polynomials in high school. A number in decimal notation such as 628 is just evaluating the polynomial $6x^2 + 2x + 8$ at $x = 10$. Notice how that example we chose just happens to be the concatenation of the first two Perfect Numbers ;). Exposing young students to different bases and having them feel comfortable with different bases prepares them for other concepts years later.

The jump to dealing with complex numbers and dividing Gaussian integers does not need to be put off until late in one's academic career. We have seen that using this device known as Exploding Dots people of all ability and knowledge levels can be dividing Gaussian integers in no time once they are exposed to this simple idea that can take them so far into joyous understanding and ownership of the ideas.

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Appendices

A Let's explore the geometric series formula

Much of the material presented in these appendices is the work of James Tanton conducted in [KB]. Let's use an $1 \leftarrow x$ machine to evaluate $\frac{1}{1-x}$. This is the very simple polynomial 1 divided by $1-x$. So we start with just one dot and are looking for the pattern of one antidot next to one dot. We surely don't see any patterns in the picture of just 1, but of course we can make it happen if we deal with the consequences. To give us our pattern we can put in a dot/antidot pair to give us one copy of what we want. Then we can do it again, and again, and again... In fact, we'll be doing this forever! Looking at 40 we see that we have one 1, and one x, and one x^2, and one x^3, and so on. So we have $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ...$ The answer is an infinite sum.

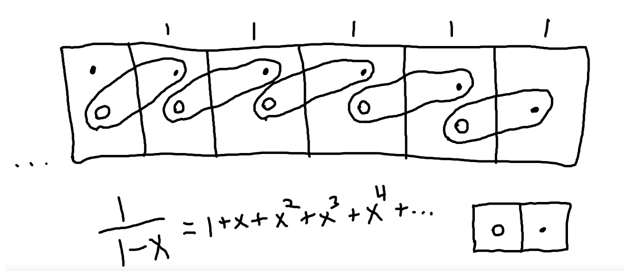

Figure 40: $\frac{1}{1-x}$

The equation we obtained here is quite a famous formula called the *geometric series formula* and is often given in many upper-level high school textbooks for students to use. However it is often written backwards of our equation with using the variable r rather than

x. $1 + r + r^2 + r^3 + ... = \frac{1}{1-r}$.

If we were in a calculus class, we could say that we just calculated the Taylor Series of the rational function $\frac{1}{1-x}$. That sounds pretty scary! But we were able to compute this playing with dots-and-boxes and it wasn't scary at all. It was actually kind of fun! But should we believe infinite sums?

We saw that the infinite sum $1+x+x^2+x^3+...$ naturally appears when we compute $\frac{1}{1-x}$. But what happens if x is let's say 2? Then the geometric series formula says $1+2+4+8+16+... = \frac{1}{1-2}$ which is -1. Well, that's just absurd! However, if $x = 0.1$, then the geometric series formula says that $1+(0.1)+(0.1)^2+(0.1)^3+... = 1+0.1+0.01+0.001+... = 1.111...$ $= \frac{1}{1-0.1} = \frac{1}{0.9} = \frac{10}{9}$ This is one and one-ninth. And we could easily use a $1 \leftarrow 10$ machine to divide 1 by 9 to see that $\frac{1}{9} = 0.111...$ as a decimal. We just take our machine and extend boxes to the right as well and then we can unexplode and look for our pattern as we do the division. In this case the geometric series formula is correct. So when can we believe this formula and when can we not?

B Should we believe infinite sums?

In a purely mechanical sense, without regards to arithmetic there is some version of truth to the claim that $1+2+4+8+... = \frac{1}{1-2}$. So if $1+2+4+8+... = \frac{1}{1-2}$, then multiplying $1+2+4+8+...$ by $1-2$ should equal 1. Does it?

$$\begin{aligned}(1-2)\times(1+2+4+8+...) &= (1-2)+(1-2)\times 2+(1-2)\times 4+(1-2)\times 8+... \\ &= 1-2+2-4+4-8+8-16+... \\ &= 1+0+0+0+... \\ &= 1\end{aligned}$$

So $1+2+4+8+...$ really does behave like $\frac{1}{1-2}$. This still isn't too satisfying as we want to know when $1+x^2+x^3+... = \frac{1}{1-x}$ is actually true as an arithmetic statement.

When we study infinite sums in calculus we learn that $1+x+x^2+x^3+... = \frac{1}{1-x}$ is true as a statement of arithmetic for small values of x, specifically for values between -1 and 1. We saw that the formula is valid for $x = 0.1$, but not for $x = 2$. Regular polynomial division shows that $\frac{1-x^2}{1-x} = 1+x$, and $\frac{1-x^3}{1-x} = 1+x+x^2$, and $\frac{1-x^4}{1-x} = 1+x+x^2+x^3$, and so on. We can see from IV.39 that if we divide $1-x^n$ by $1-x$ we will always have an antidot

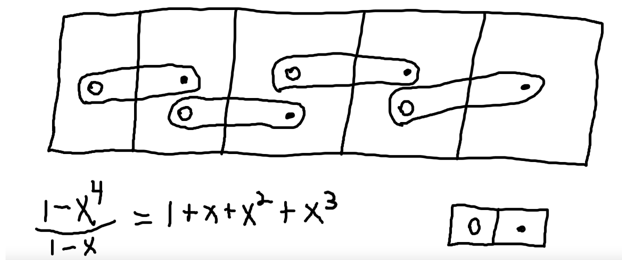

Figure 41: $\frac{1-x^4}{1-x}$

in our leftmost box and a dot in our rightmost box. So we can always keep creating our pattern of $1-x$. In general, we see that $1+x+x^2+...+x^{n-1} = \frac{1-x^n}{1-x}$. And as x gets larger and larger it looks like we're getting the infinite geometric sum. So the question becomes: what is the limit of $\frac{1-x^n}{1-x}$ as n keeps getting larger and heads off to infinity. This depends on whether or not x^n has a limit value as n continues to grow. We know that x^n gets closer and closer to zero as n grows for any value of x between -1 and 1. So for $-1 < x < 1$, we have $1+x+x^2+x^3+... = \frac{1-0}{1-x} = \frac{1}{1-x}$. The geometric series formula can thus be believed as a statement of arithmetic for $-1 < x < 1$, at least. But let's take a look at another approach.

C What must $1 + x + x^2 + ...$ equal IF it is meaningful?

Another approach to examining the geometric series formula is to assume that the infinite sum $1 + x + x^2 + x^3 + ...$ is meaningful and has an answer. We will call this answer S. Then

$$\begin{aligned} S &= 1 + x + x^2 + x^3 + ... \\ &= 1 + x(1 + x + x^2 + ...) \\ &= 1 + xS \end{aligned}$$

So we have $S = 1 + xS$

$$\begin{aligned} S - xS &= 1 \\ S(1 - x) &= 1 \end{aligned}$$

From which we get $S = \frac{1}{1-x}$.

This gives us the following theorem:

Theorem 3.1. *IF the infinite sum $1 + x + x^2 + ...$ has an answer, then that answer must be $\frac{1}{1-x}$.*

This theorem makes no assertion as to whether or not the infinite sum is meaningful and has an answer in the first place. Our approach with Exploding Dots likewise proves that IF $1 + x + x^2 + x^3 + ...$ is meaningful to you, then the sum must be $\frac{1}{1-x}$. One must decide if this infinite sum is meaningful. In the world of calculus it is often said to be meaningful if $-1 < x < 1$.

Could there possibly be other systems of arithmetic that offer other meanings? Exploding Dots can also be utilized to explore adic number systems! The statement $1+2+4+8+... = -1$ seems to be meaningless in our normal way of doing arithmetic. But who says we can't look at things in a different way? When we think of integers on a number line, we tend to think of them spaced apart additively. That is, if we start at 1 and add two steps we'll be at 3. Then if we added four steps we'd be at position 7 and so on. In this viewpoint $1 + 2 + 4 + 8 + ...$ takes us infinitely far to the right on the number line. We could say $1+2+4+8+... = \infty$, it certainly doesn't equal -1. But what if we thought of numbers multiplicatively rather than additively? In particular, let's think of multiples of powers of two since we are focusing on the sum $1 + 2 + 4 + 8 + ...$ Now 0 is the most divisible number of all with regard to being divisible by 2. Zero can be divided by 2 once, or twice. or three times. In fact, we can divide 0 by two as many times as we like and keep dividing.

With regard to two-ness, we can conclude that 8 is somewhat zero-like: we can divide it by two three times and still get an integer. But 32 is more zero-like as we can divide it by 2 five times. And 2^{100} is even more zero-like. So if we think in this context then we have,

$$\begin{aligned} 1 + 2 = 3 = 4 - 1 \\ 1 + 2 + 4 = 7 = 8 - 1 \\ 1 + 2 + 4 + 8 = 15 = 16 - 1 \\ . \\ . \\ . \\ 1 + 2 + ... + 2^{99} = 2^{100} - 1 \end{aligned}$$

These finite sums grow to become "a number very close to zero, minus one." If we take the limit, the infinite sum therefore has value $0 - 1 = -1$. So in this multiplicative view of arithmetic, $1 + 2 + 4 + 8 + ...$ is a meaningful quantity and indeed it does have value of -1. So the geometric series formula is meaningful and correct for $x = 2$ in this context. And our work with Exploding Dots shows clearly what the answer to many infinite sums must be, IF the infinite sum has meaning to you.